

# On minimal degree of permutation groups

- $G \leq \text{Sym}(\Omega)$

e.g.  $\Omega = \{1, 2, 3, 4, 5\}$ ,  $G = S_5, A_5, D_{10}$

- for  $x \in G \leq \text{Sym}(\Omega)$ ,

$\text{deg}(x)$ : number of points moved by  $x$

$\text{fix}_\Omega(x)$ :  $\{\omega \in \Omega \mid \omega^x = \omega\}$

e.g.  $x = (1, 2, 3) \in \text{Sym}(\{1, 2, 3, 4, 5\})$ ,  $\text{deg}(x) = 3$ ,  $\text{fix}_\Omega(x) = \{4, 5\}$

- **minimal degree**

minimal degree  $\mu(G)$ :  $\min \{\text{deg}(x) : x \in G\}$

e.g.  $\mu(\text{Sym}(n)) = 2$ ,  $\mu(\text{Alt}(n)) = 3$  (for  $n \geq 3$ )

- $G_\omega := \{g \in G \mid \omega^g = \omega\}$

ex:  $G = S_n, G_1 = S_{n-1}$

The action of  $G$  on  $\Omega$  and  $[G:G_\omega]$  are equivalent

- **block:**  $B \subseteq \Omega$  with either  $B^g = B$  or  $B^g \cap B = \emptyset$  for any  $g \in G$

**block system:** for finite  $\Omega$ ,  $\Omega = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k, B_i^g = B_j$

ex:  $\{1, 2, 3, 4\} = \{1, 2\} \sqcup \{3, 4\}, G = \langle (1, 3, 2, 4), (1, 2) \rangle \cong D_8$

block systems of  $G \leftrightarrow$  subgroups of  $G$  containing  $G_\omega$

$G$  is **primitive** on  $\Omega$  if there is no **non-trivial** block system of  $G$  acting on  $\Omega$

- $G$  is primitive  $\iff G_\omega$  is a maximal subgroup of  $G$

Aschbacher 1989: Classification of primitive actions of classical groups

O'Nan-Scott-Preager: Classification of finite primitive  $G$  (90s)

History for  $\mu(G)$ ,  $G$  primitive

Babai 1981: If  $\text{Alt}(\Omega) \not\leq G$ , then  $\mu(G) \geq 1/2(\sqrt{|\Omega|} - 1)$

Liebeck-Saxl 1990:

- either 1.  $\mu(G) \geq \frac{1}{2}|\Omega|$ 
  2.  $A_m^r \trianglelefteq G \leq S_m^r \rtimes S_r \leq \text{Sym}(\Delta^r)$ ,  $\Delta = \{k\text{-subsets of } [m]\}$  or  $\Delta = [6]$
  3.  $L^r \trianglelefteq G \trianglelefteq L_1^r \rtimes S_r \leq \text{Sym}(\Delta^r)$ ,  $L$  is a simple group of Lie type over  $\mathbb{F}_2$ ,  
 $L = \text{soc}(L_1)$ ,  $L_1$  is primitive on  $\Delta$

Guralnick-Magaard 1998:

- if  $G$  is primitive with  $\mu(G) < \frac{1}{2}|\Omega|$ , then
  - either 1.  $G$  is in LS.2 with  $k \leq m/4$ ,  $m = 6$
  - or 2.  $G$  is in LS.3 with  $L_1 = O_m(2)$  with  $m \geq 6$

Observations:

1. for  $x \in G \leq \text{Sym}(\Omega)$ ,  $\text{fix}_\Omega(x) \subseteq \text{fix}_\Omega(x^m)$
2. for  $x \in G$ , and any block system  $\mathcal{B}$

$$\Omega = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k$$

$$|\text{fix}_\Omega(x)| / |\Omega| \leq |\text{fix}_{\mathcal{B}}(x)| / |\mathcal{B}|$$

3.  $\text{deg}(x) / |\Omega| = 1 - |\text{fix}_\Omega(x)| / |\Omega|$

**fixed-point-ratio** of  $x$  with respect to the action of  $G$  on  $\Omega$  is

$$|\text{fix}_\Omega(x)| / |\Omega|$$

Therefore,

$\min \{ \text{deg}(x) / |\Omega| \mid x \in G \} \iff$  maximum fixed-point-ratio among

all **prime-order** elements of  $G$  respect to all **primitive** actions

$\mathbb{F}_q$ : finite field,  $q = p^f$  for prime  $p$

$\mathrm{GL}_n(q)$ : invertible  $n \times n$  matrices over  $\mathbb{F}_q$

$\mathrm{SL}_n(q)$ :  $n \times n$  matrices of determinant 1 over  $\mathbb{F}_q$

$\mathrm{PSL}_n(q)$ :  $\mathrm{SL}_n(q) / Z(\mathrm{SL}_n(q))$

ex:  $\mathrm{GL}_n(2) = \mathrm{SL}_n(2) = \mathrm{PSL}_n(2)$

elements in  $\mathrm{GL}_n(q)$

- $x \in \mathrm{GL}_n(q)$  is similar to a Jordan canonical form over  $\overline{\mathbb{F}_q}$

$|x|$  is coprime to  $p \iff x$  is diagonalizable over  $\overline{\mathbb{F}_q}$ ;

$|x|$  is a power of  $p \iff$  all eigenvalues of  $x$  are 1

Elements of a prime order  $r$ ,  $r \neq p$

$x \in \text{GL}_n(q)$ ,  $|x| = r \iff$  eigenvalues of  $x$  are  $r$ -th root of unity, and  $x \neq 1$

- Take a field  $\mathbb{K} = \mathbb{F}_{q^i}$  s.t (TFAE)

$\mathbb{K}$  is the splitting field of  $x^r - 1$  over  $\mathbb{F}_q$ ,

$\mathbb{K} = \mathbb{F}_q[\omega]$ , where  $\omega$  is an  $r$ -root of unity

$i$  is the minimal integer such that  $r$  divides  $q^i - 1$

- $x \in \text{GL}_n(q) \iff x_{ij}^q = x_{ij}$

the eigenvalues of  $x$  are invariant under  $\text{Gal}(\mathbb{F}_{q^i}/\mathbb{F}_q)$

$x$  is similar to  $\text{diag}(\Lambda_1, \dots, \Lambda_k, I_e)$  over  $\mathbb{F}_q$  with

1. the characteristic polynomial of  $\Lambda_j$  is irreducible over  $\mathbb{F}_q$ ,

with eigenvalues  $\{\omega, \omega^q, \dots, \omega^{q^{i-1}}\}$ , where  $\omega \neq 1, \omega^r = 1$

2.  $\Lambda_i$  is similar to the  $\mathbb{F}_q$ -linear map of  $\mathbb{F}_{q^i}$  by multiplying by  $\omega$

$$\mathbb{F}_{q^i} \longrightarrow \mathbb{F}_{q^i}: \lambda \longmapsto \lambda\omega$$

## Guralnick-Magaard induction

$H \leq \text{Sym}(\Omega)$ ,  $[H, H]$  is quasisimple

$\mu_H(g)$  maximum fixed-point-ratio of  $g$  among all faithful action of  $H$

- **key 1:**  $H = A \times B$ ,  $g = (a, b) \in H$ ,  $[B, B]$  quasi-simple with  $b \notin Z([B, B])$

either  $|\text{fix}_\Omega(g)|/|\Omega| \leq \mu_B(b)$

or  $[B, B] \leq G_\omega$

- **key 2:**  $P = U \rtimes L$ ,  $U \cong \mathbb{F}_p^k$  ( $U$  elementary abelian of order  $p$ )

$[L, L]$  quasi-simple,  $[L, L]$  acts irreducibly on  $U$ ,  $g \in P$  s.t.  $gU \notin Z(P/U)$

then for an action of  $P$  on  $\Omega$ , either

1.  $|\text{fix}_\Omega(g)|/|\Omega| \leq \mu_{P/U}(gU)$

2. for some  $u \in U$ , we have  $C_L(u) \leq P_\omega$

Example  $G = L_n(2)$  by induction

Case 1:  $\dim(C_V(g)) \geq 1$ , so  $g \in P_1$

$P_1 = U \rtimes L$  (the stabilizer of 1-dim subspace)

$$U = \left( \begin{array}{c|c} 1 & * \\ \hline 0 & I_{n-1} \end{array} \right) \cong \mathbb{F}_2^{n-1}, \quad L \cong \left\{ \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right) \mid A \in L_{n-1}(2) \right\} \cong L_{n-1}(2)$$

Case 1.1 If  $g$  is similar to an element in  $L$  over  $\mathbb{F}_q$

(conjugate in  $G = \mathrm{GL}_n(2)$  to an element in  $L$ )

then by induction  $|\mathrm{fix}_\Omega(g)|/|\Omega| < 1/2$

Case 1.2. If  $g$  is not conjugate in  $G$  to any element in  $L$

then  $|g| = 2$ ,

$g$  is similar to  $\mathrm{diag}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \dots, \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)\right)$  (since  $|g| = 2$ )

so by key 2, either 1.  $|\mathrm{fix}_\Omega(g)|/|\Omega| \leq \mu_{P_1/U}(gU)$

or  $G_\omega$  contains  $C_L(u)$  for some  $u \in U$ ,

note that  $P = U \rtimes L$ , with  $U \cong \mathbb{F}_2^{n-1}$  and  $L$  acts transitively on

the non-zero elements in  $U$ , so WMA  $u = \left( \begin{array}{c|c|c} 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right)$ , and

$$C_L(u) = \left\{ \left( \begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & \alpha' & A \end{array} \right) \mid \alpha' \in \mathbb{F}_2^{n-2}, A \in L_{n-2}(2) \right\} \cong \mathbb{F}_2^{n-2} \rtimes L_{n-2}(2)$$

Subgroups generate by transvections:

J. McLaughlin 1989: If  $\mathbb{K} \neq \mathbb{F}_2$ ,  $\dim(V) > 2$ ,  $G$  is a subgroup generated by transvections, and 1 is the only unipotent subgroup normalized by  $G$ , then

$G = G_1 \times \dots \times G_t$  correspond to  $V = V_1 \oplus \dots \oplus V_t$  with

1.  $V_i$  invariant under  $G$
2.  $G_i|_{V_j} = 1$  for  $i \neq j$ , 3.  $G_i|_{V_i} = \text{SL}(V_i)$  or  $\text{Sp}(V_i)$

P.J Cameron 1989: for  $\mathbb{K} = \mathbb{F}_2$  two more cases

1.  $G_i|V_i = O(V_i)$
2.  $G_i|V_i \cong \text{Sym}(\Omega)$  with  $|\Omega| \geq 5$ ,  $V_i$  is the irreducible permutation submodule of  $G_i$

Now if  $G_\omega \neq P_{n-1}, P_1$  (the stabilizer of  $(n-1)$ -dimensional subspace)

Since  $G_\omega$  contain  $C_L(u)$ , the unipotent subgroup normalized by  $C_L(u)$  is

$$\left( \begin{array}{c|c} I_2 & 0 \\ \hline * & I_{n-2} \end{array} \right),$$

$G_\omega = P_{n-2}$ , thus  $G_\omega = P_1, P_{n-1}, P_2, P_{n-2}$

Now  $g \sim \text{diag}\left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \dots, \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)\right)$  so

1.  $\dim(C_V(g)) = n/2$ , there are  $2^{n/2} - 1$  1-dim spaces fixed by  $g$

and there are  $(2^{n/2} - 1)(2^{n/2-1} - 1)/3$  2-dim s.t  $g$  acts trivially on them

2. as a permutation on  $\mathbb{F}_2^n$ ,  $g$  fixes  $2^{n/2}$  vectors and acts as  $2^{n/2-1}(2^{n/2} - 1)$

2-cycles on remaining vectors

$\{2\text{-dim subspaces s.t } g \text{ acts as a transvection}\} \leftrightarrow \{2\text{-cycles of } g\}$

so there are  $2^{n/2-1}(2^{n/2} - 1)$  2-dim spaces s.t  $g$  acts as a transvection

- Totally there are  $(2^n - 1)(2^{n-1} - 1) / 3$  2-subspace of  $\mathbb{F}_2^n$

$$\text{so } |\text{fix}_\Omega(g)| / |\Omega| \leq \frac{3 \times 2^{n/2-1}(2^{n/2} - 1)}{(2^n - 1)(2^{n-1} - 1)} + \frac{(2^{n/2} - 1)(2^{n/2-1} - 1)}{(2^n - 1)(2^{n-1} - 1)} < \frac{1}{2}$$

## Case 2.

- $g$  acts irreducibly on  $V$

so  $g \sim [\Lambda]$ , the eigenvalues of  $g$  over  $\mathbb{F}_{2^n}$  are  $\{\omega, \omega^2, \dots, \omega^{2^n-1}\}$

with  $\omega$  primitive  $r$ -th root of unity and  $\mathbb{F}_2[\omega] = \mathbb{F}_{2^n}$

- The centralizer of  $g$

$\mathbb{F}_{2^n} \cong \mathbb{F}_2^n$  as an  $\mathbb{F}_2$ -linear space, multiply by any element in  $\mathbb{F}_{2^n}$  is a  $\mathbb{F}_2$ -linear map

so  $\mathbb{F}_{2^n}^\times \leq C_{L_n(2)}(g)$ , and  $C_{L_n(2)}(g) \cong \mathbb{F}_{2^n}^\times$  (called singer cycle)

- **Permutation character:**  $\pi(g) = \text{fix}_\Omega(g)$

$$|\text{fix}_\Omega(g)|/|\Omega| \leq \frac{1}{m+1} + \frac{|C_{L_n(2)}(g)|^{1/2}}{m},$$

with  $m$  the minimum degree among all complex characters

**Case 3:**  $g \sim [\Lambda_1, \dots, \Lambda_{n/d}]$  with  $\dim(\Lambda_i) = d$

for  $n \leq 5$  by GAP with permutation character, one exceptional case

$G = L_4(2) \cong A_8$ , with  $G_\omega = A_7$  (the action is on 8 pts)

for  $n \geq 5$ ,  $g$  is conjugate to an element in  $L_d(2) \times L_{n-d}(2)$

$$\left\{ \left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \mid A \in L_d(2), B \in L_{n-d}(2) \right\}$$

By [key 1](#) and induction either 1.  $|\text{fix}_\Omega(g)|/|\Omega| < \mu_{L_{n-d}}(g|_{L_{n-d}(2)})$

or 2.  $G_\omega$  contains  $L_{n-d}(2)$

In 1, either  $|\text{fix}_\Omega(g)|/|\Omega| < 1/2$ , or  $n = 6, d = 2$ ,  $G_\omega$  contains  $A_7$

In 2, the unipotent subgroup normalized by  $\langle g, L_{n-d}(2) \rangle$  is

$$\left( \begin{array}{c|c} I_d & 0 \\ \hline * & I_{n-d} \end{array} \right), \text{ or } \left( \begin{array}{c|c} I_d & * \\ \hline 0 & I_{n-d} \end{array} \right), \text{ or } 1$$

so  $G_\omega = P_d, P_{n-d}$  (stabilizer of  $d$  or  $n-d$ -dim subspaces)

or  $d = n-d$  and  $G_\omega = L_d(2)^2 \rtimes S_2$

So  $G_\omega = P_d, P_{n-d}, L_d(2)^2 \rtimes S_2$ , or  $n = 6$  and  $G_\omega = \text{Sp}(6, 2)$

$g \sim [\Lambda_1, \dots, \Lambda_{n/d}]$  with the characteristic polynomial of  $\Lambda_i$  irreducible

over  $\mathbb{F}_2$ , with eigenvalues in  $\mathbb{F}_{2^d} \setminus \{\omega, \omega^2, \dots, \omega^{2^d-1}\}$ , and  $\mathbb{F}_2[\omega] = \mathbb{F}_{2^d}$

$\Lambda_i$  similar to multiplying by  $\omega$  on  $\mathbb{F}_{2^d}$

- For any  $d$ -dim subspace  $W$  fixed by  $g$ ,  $g|_W = \Lambda_i$

so when  $g \sim [\Lambda, \dots, \Lambda]$ ,  $g$  fixes most  $d$ -dim subspaces

i.e  $g$  is similar to a scalar matrix in  $\text{GL}_{n/d}(2^d)$ ,  $g$  fixes  $\frac{2^n - 1}{2^d - 1}$

$d$ -dim subspaces

- Similarly, when  $g$  is similar to a scalar matrix in  $\text{GL}_{n/d}(2^d)$ ,  $g$

fixes most  $n - d$  dim subspaces

when  $n = 2d$  and  $g$  is similar to a scalar matrix in  $\text{GL}_2(2^d)$ ,

then  $g$  fixes most direct decomposition of  $V$  to 2  $d$ -dim subspaces

If  $n = 6, d = 2$  ( $G = L_6(2)$ ) and  $G_\omega = \text{Sp}(6, 2)$

Then the action of  $G$  on  $\Omega$  is equivalent to an orbit of  $G$  acting

on  $\Lambda^2(V) = \text{Span}_{\mathbb{F}_2}\{e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i, j \leq 6\}$

So  $|\text{fix}_\Omega(g)| = |C_{\Lambda^2(V)}(g) \setminus \{0\}|$

$g \sim [\omega, \omega^2, \omega, \omega^2, \omega, \omega^2]$  over  $\mathbb{F}_4$ , with  $\omega^3 = 1$

Take  $\{e_1, \dots, e_6\}$  eigenvectors of  $g$  over  $\mathbb{F}_4 \otimes_{\mathbb{F}_2} \mathbb{F}_2^6 = \mathbb{F}_4^6$  then

$C_{\mathbb{F}_4 \otimes_{\mathbb{F}_2} \Lambda^2(V)}(g)$

$= \text{Span}_{\mathbb{F}_4}\{e_1 \wedge e_2, e_1 \wedge e_4, e_1 \wedge e_6, e_3 \wedge e_2, e_3 \wedge e_4, e_3 \wedge e_6, e_5 \wedge e_2, e_5 \wedge e_4, e_5 \wedge e_6\}$

and  $\dim(C_{\Lambda^2(V)}(g)) = \dim(C_{\mathbb{F}_4 \otimes_{\mathbb{F}_2} \Lambda^2(V)}(g)) = 9$

So  $|\text{fix}_\Omega(g)|/|\Omega| = \frac{2^9 - 1}{[G:G_\omega]}$

**Burness Guranick 2022:** 1.  $(G, x, \Omega)$  with  $\text{fix}_\Omega(x) > \frac{1}{r+1}$  2. lists of  $G$  with  $\mu(G) < \frac{2}{3}|\Omega|$

